

# Robust pole placement with Moore's algorithm

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## Abstract

We consider the classic problem of pole placement by state feedback. We adapt the Moore eigen-structure assignment algorithm to obtain a novel parametric form for the pole-placing gain matrix, and introduce an unconstrained nonlinear optimization algorithm to obtain a gain matrix that will deliver robust pole placement. Numerical experiments indicate the algorithm's performance compares favorably against several other notable robust pole placement methods from the literature.

## I. INTRODUCTION

We consider the classic problem of pole placement for LTI systems in state space form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

where, for all  $t \in \mathbb{R}$ ,  $x(t) \in \mathbb{R}^n$  is the state, and  $u(t) \in \mathbb{R}^m$  is the control input.  $A$  and  $B$  are appropriate dimensional constant matrices. We assume that  $B$  has full column rank. We let  $\mathcal{L} = \{\lambda_1, \dots, \lambda_\nu\}$  be a self-conjugate set of  $n$  complex numbers, with associated algebraic multiplicities  $\mathcal{M} = \{m_1, \dots, m_\nu\}$  satisfying  $m_1 + \dots + m_\nu = n$ . The problem of *exact pole placement by state feedback* (EPP) is that of finding a real matrix  $F$  such that the closed-loop matrix  $A + BF$  has non-defective eigenvalues in  $\mathcal{L}$ , i.e  $F$  satisfies

$$(A + BF)X = X\Lambda \quad (2)$$

where  $\Lambda$  is a  $n \times n$  diagonal matrix obtained from the eigenvalues of  $\mathcal{L}$ , including multiplicities, and  $X$  is a non-singular matrix of closed-loop eigenvectors of unit length. If  $(A, B)$  has any uncontrollable modes, these are assumed to be included within the set  $\mathcal{L}$ . The EPP problem has been studied for several decades, and the existence of such a matrix yielding diagonal  $\Lambda$  requires the  $m_i$  to satisfy certain inequalities in terms of the controllability indices of the pair  $(A, B)$  [2];

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in particular  $m_i \leq m$  for all  $m_i \in \mathcal{M}$  is required. In this paper we shall assume  $(A, B, \mathcal{L}, \mathcal{M})$  are such that at least one  $F$  exists that yields diagonal  $\Lambda$ . Notable early papers offering algorithms to obtaining the required gain matrix  $F$  include [3], which gave a method for single-input single output (SISO) system, but this was often found to be numerically inaccurate. Varga [4] gave a numerically reliable method to obtain  $F$  for multiple-input multiple output (MIMO) systems.

For SISO systems,  $F$  is unique, while for MIMO systems it is not, and this naturally invites the selection of  $F$  that achieves the desired pole placement and also possesses other desirable characteristics, such as minimizing the control input amplitude used, and improving numerical stability. In order to consider optimal selections for the gain matrix, it is important to have a parametric formula for the set of gain matrices that deliver the desired pole placement, and numerous such parameterizations have appeared. Bhattacharyya and de Souza [5] gave a procedure for obtaining the gain matrix by solving a Sylvester equation in terms of a  $n \times m$  parameter matrix, provided the closed-loop eigenvalues did not coincide with the open loop ones. Fahmy and O'Reilly [6], gave a parametric form in terms of the inverses of the matrices  $A - \lambda_i I$ , which also required the assumption that the closed loop eigenvalues were all distinct from the open loop ones. Kautsky *et al* [7] gave a parametric form involving a QR-factorization for  $B$  and a Sylvester equation for  $X$ ; this formulation did not require the closed-loop poles to be different from the open-loop poles.

The classic eigenstructure assignment algorithm of B.C. Moore [9] quantified the freedom to simultaneously assign both the closed-loop eigenvalues, and also select the associated eigenvectors. As such it implicitly solved the EPP problem, but it did not explicitly provide a parametric formula for the pole-placing matrix, nor did it address any optimal pole placement problem. In this paper we adapt Moore's algorithm to obtain a simple parametric formula for the pole-placing gain matrix, in terms of an  $n \times m$  parameter matrix. The method obtains the eigenvector matrix  $X$  by selecting eigenvectors from the nullspaces of the system matrices, and thus avoids the need for coordinate transformations.

The *robust exact pole placement problem* (REPP) involves solving the EPP problem and also obtaining  $F$  that renders the eigenvalues of  $A + BF$  as insensitive to perturbations in  $A$ ,  $B$  and  $F$  as possible. Numerous results [10] have appeared linking the sensitivity of the eigenvalues to various measures of the conditioning of  $X$ , in terms of the Euclidean and Frobenius norms. This classic optimal control problem also has an extensive literature, and typically two approaches

have been used to obtain good robust conditioning.

Perhaps the best-known method for the REPP is that of Kautsky *et al* [7], which involved selecting an initial candidate set of closed-loop eigenvectors and then using a variety of heuristic methods to make these vectors more orthonormal. This method has been implemented as MATLAB®'s *place* command; this implementation includes a heuristic extension to accommodate complex conjugate pairs in  $\mathcal{L}$ . This algorithm is also the basis of MATHEMATICA®'s *KNVD* command. The use of the *place* algorithm has become wide-spread in the control systems literature, and introductory texts advocating its use include [11] and [12], among many others.

Since the publication of [7], many alternative methods have been proposed for the REPP. Tits and Yang [13] revisited the heuristic methods of [7] and offered a range of improvements; the algorithms were shown to be globally convergent. Byers and Nash [14], Tam and Lam [15] and Varga [16] cast the problem as an unconstrained nonlinear optimization problem, in terms of the Frobenius conditioning, to be solved by gradient iterative search methods. [17] introduced a method for minimizing the 'departure from normality' robustness measure, which considers the size of the upper triangular part of the Schur form. Ait Rami *et al* [18] introduced a global constrained nonlinear optimal problem in terms of a Sylvester equation and showed that the solution could be approximated by a convex linear problem for which the authors gave an LMI-based algorithm.

Various authors have provided surveys comparing the performance of several of these algorithms. Sima *et al* [19] conducted testing of the algorithms from [4], [7] and [13] on collections of systems of varying dimensions; they concluded that the method of [13] generally gave superior Euclidean (2-norm) conditioning and also improved accuracy. [17] considered the eleven benchmark systems in the Byers-Nash collection (see Section IV for a discussion of this collection), and compared the author's proposed methods, based on the Schur form of the open loop systems, with those of [7] and [13] against a range of robustness measures. The methods of [17] generally gave inferior results to those of [7] and [13], with respect to the Frobenius conditioning. [18] tabulated figures results for the Frobenius conditioning performance of methods [7], [13], [14] and [16]. However, the conditioning values were compiled directly from these papers. Since some of these methods were introduced into the literature more than two decades ago, and noting that computational resources have improved dramatically over this time, using values from original publications may unfairly disadvantage the earlier methods, in particular [14].

In this paper we add to this extensive literature in several ways. In Section 2 we introduce our parametric form for the pole-placing gain matrix that solves the EPP. The formula is an adaptation of the pole placement method of Moore [9]; the novelty here is to use Moore's method to obtain a parametric formula for both  $X$ , the matrix of eigenvectors and  $F$ , the pole-placing gain matrix. We further show the parametric form is comprehensive, in that it generates all possible  $X$  and  $F$  that solve (2), for the case where the eigenvalues have multiplicity of at most  $m$ . In Section 3 we utilize this parametric form to propose an unconstrained optimization problem to seek solutions to the REPP, to be solved by gradient search methods. Our approach most closely resembles that of [14], but with a different parametric formulation for the pole-placing gain matrix.

In Section 4 we select five of the most prominent methods for the REPP [7], [13], [16], [14] and [18], and conduct extensive numerical testing to compare their performance against our method. The first three of these were chosen as they are widely used in the forms of the MATLAB<sup>®</sup> toolboxes *place*, *robust* and *sylvplace* respectively. [14] has attracted a large number of citations over more than two decades, and [18] is the most recent publication to offer a novel approach for the REPP. All methods were implemented in MATLAB<sup>®</sup>2012a, running on the same computing platform. In addition to conditioning, we also compare their accuracy, matrix gain and runtime. Finally, Section 5 offers some conclusions as to the relative performance of these six methods; our method will be shown to offer some performance advantages over all the other methods surveyed.

## II. POLE PLACEMENT VIA MOORE'S ALGORITHM

We now revisit Moore's method [9] and adapt it to give a simple parametric formula for a gain matrix  $F$  that solves the pole placement problem, in terms of an arbitrary real parameter matrix. We begin with some definitions and notation. For each  $i \in \{1, \dots, \nu\}$ , we define the  $n \times (n + m)$  system matrix

$$S(\lambda_i) = [A - \lambda_i I_n \ B] \quad (3)$$

where  $I_n$  is the identity matrix of size  $n$ . We let  $T_i$  be a basis matrix for the nullspace of  $S(\lambda_i)$ , we use  $s_i$  to denote the dimension of this nullspace, and we denote  $T = [T_1 \dots T_\nu]$ . It follows that  $s_i = m$ , unless  $\lambda_i$  is an uncontrollable mode of the pair  $(A, B)$ , in which case we will have  $s_i > m$ . Let  $M$  denote any complex matrix partitioned into submatrices  $M = [M_1 | \dots | M_\nu]$

such that any complex submatrices occur consecutively in complex conjugate pairs. We define a real matrix  $Re(M)$  of the same dimension as  $M$  thus: if  $M_i$  and  $M_{i+1}$  are consecutive complex conjugate submatrices of  $M$ , then the corresponding submatrices of  $Re(M)$  are  $\frac{1}{2}(M_i + M_{i+1})$  and  $\frac{1}{2j}(M_i - M_{i+1})$ . Finally, for any real or complex matrix  $X$  of with at least  $n + m$  rows, we define matrices  $\bar{\pi}(X)$  and  $\underline{\pi}(X)$  by taking the first  $n$  and last  $m$  rows of  $X$ , respectively.

*Proposition 2.1:* Let the eigenvalues  $\{\lambda_1, \dots, \lambda_\nu\}$  be ordered so that, for some integer  $s$ , the first  $2s$  values are complex while the remaining are real, and for all odd  $i \leq 2s$  we have  $\lambda_{i+1} = \bar{\lambda}_i$ . Let  $K := \text{diag}(K_1, \dots, K_\nu)$ , where each  $K_i$  is of dimension  $s_i \times m_i$ , and for all odd  $i \leq 2s$ , we have  $K_i = \bar{K}_{i+1}$ . Let  $M(K)$  be an  $(n + m) \times n$  complex matrix given by

$$M(K) = TK \quad (4)$$

and let

$$X(K) = \bar{\pi}(M(K)), \quad (5)$$

$$V(K) = \bar{\pi}(Re(M(K))) \quad (6)$$

$$W(K) = \underline{\pi}(Re(M(K))) \quad (7)$$

For almost every choice of the parameter matrix  $K$ , the rank of  $X$  is equal to  $n$ . The set of all  $m \times n$  gain matrices  $F$  satisfying (2) is parameterised in  $K$  as

$$F(K) = W(K)V(K)^{-1} \quad (8)$$

where  $K$  is such that  $\text{rank}(X(K)) = n$ .

*Proof:* For any given  $K$ , let  $M(K)$  be partitioned according to

$$M(K) = \begin{bmatrix} V'_1 & \dots & V'_\nu \\ W'_1 & \dots & W'_\nu \end{bmatrix} \quad (9)$$

where each  $V'_i$  and  $W'_i$  are matrices of dimensions  $n \times m_i$  and  $m \times m_i$  respectively, such that

$$(A - \lambda_i I_n)V'_i + BW'_i = 0 \quad (10)$$

Note that, for odd  $i \leq 2s$ , we have that  $V'_i = \bar{V}'_{i+1}$  are conjugate matrices, as  $K_i = \bar{K}_{i+1}$ . Moreover, since  $\mathcal{L}$  is symmetric, we also have  $m_i = m_{i+1}$ . Define real matrices

$$V_i = \begin{cases} \frac{1}{2}(V'_i + V'_{i+1}) & \text{if } i \leq 2s \text{ is odd,} \\ \frac{1}{2j}(V'_{i-1} - V'_i) & \text{if } i \leq 2s \text{ is even,} \\ V'_i & i > 2s \end{cases} \quad (11)$$

and define  $W_i$  similarly. Then matrices  $X$ ,  $V$  and  $W$  in (5)-(7) may be written as

$$X = [V'_1 \ V'_2 \ \dots \ V'_{2s} | V'_{2s+1} \ V'_{2s+2} \ \dots \ V'_\nu], V = [V_1 \ V_2 \ \dots \ V_{2s} | V_{2s+1} \ V_{2s+2} \ \dots \ V_\nu],$$

and  $W = [W_1 \ W_2 \ \dots \ W_{2s} | W_{2s+1} \ W_{2s+2} \ \dots \ W_\nu]$ . Let

$$R_i = \frac{1}{2} \begin{bmatrix} I_{m_i} & -jI_{m_i} \\ I_{m_i} & jI_{m_i} \end{bmatrix} \quad (12)$$

Then for each odd  $i \leq 2s$ , we have  $[V'_i \ V'_{i+1}]R_i = [V_i \ V_{i+1}]$  and  $[W'_i \ W'_{i+1}]R_i = [W_i \ W_{i+1}]$ . Now assume  $K$  is such that  $\text{rank}(X(K)) = n$ ; then  $V(K)$  is non-singular, and we can obtain  $F$  in (8). We obtain  $F[V'_i \ V'_{i+1}] = [W'_i \ W'_{i+1}]$  for odd  $i \in \{1, \dots, 2s\}$  and  $FV'_i = W'_i$  for all  $i \in \{2s+1, \dots, \nu\}$ . Hence (10) can be written as

$$(A + BF) \begin{bmatrix} V'_i & V'_{i+1} \end{bmatrix} = \begin{bmatrix} V'_i & V'_{i+1} \end{bmatrix} \text{diag}(\lambda_i I_{m_i}, \lambda_{i+1} I_{m_i}), \text{ for odd } i \in \{1, \dots, 2s\} \quad (13)$$

$$(A + BF)V'_i = V'_i(\lambda_i I_{m_i}), \text{ for } i \in \{2s+1, \dots, \nu\}, \quad (14)$$

Thus we obtain (2). To see that this formula is comprehensive, we let  $F$  be any real gain matrix satisfying (2). The nonsingular eigenvector matrix  $X$  is comprised of column vectors  $V'_i$  of dimension  $n \times m_i$  corresponding to each eigenvalue, such that (13) and (14) hold. Applying  $F[V'_i \ V'_{i+1}] = [W'_i \ W'_{i+1}]$  for odd  $i \in \{1, \dots, 2s\}$  and  $FV'_i = W'_i$  for all  $i \in \{2s+1, \dots, \nu\}$ , we obtain  $V'_i$  and  $W'_i$  such that (10) holds. Thus each column vector of the matrix  $[V'_i \ W'_i]^T$  lies in the kernel of  $S(\lambda_i)$ , and we have a coefficient vector  $K_i$  such that  $[V'_i \ W'_i]^T = T_i K_i$ . The complex conjugacy of  $V'_i$  and  $V'_{i+1}$ , for each odd  $i \in \{1, \dots, 2s\}$ , implies the conjugacy of  $K_i$  and  $K_{i+1}$ . Thus we obtain  $M(K)$  in (4) yielding  $F$  in (8).

Finally we let  $K$  be arbitrary parameter matrix and consider the rank of  $X(K)$ . We introduce  $\Phi = \bar{\pi}(T)$  and denote  $\Phi_1, \dots, \Phi_\nu$  as a basis for  $\text{im } \Phi$ . If  $\text{rank}(X(K))$  is smaller than  $n$ , then one column of the matrix  $[\Phi_1 K_{1,1} \dots \Phi_\nu K_{\nu, m_\nu}]$  is linearly dependent of all the remaining ones. (Here we have used  $K_{i,j}$  to denote the  $j$ -th column of  $K_i$ ). For brevity, let us assume this is the last column. Then there exist  $n-1$  coefficients  $\alpha_{1,1}, \dots, \alpha_{\nu, m_\nu-1}$  (not all equal to zero) for which

$$\Phi_\nu K_{\nu, m_\nu} = \sum_{i=1}^{\nu-1} \sum_{j=1}^{m_i} \alpha_{i,j} \Phi_i K_{i,j} + \sum_{j=1}^{m_\nu-1} \alpha_{\nu,j} \Phi_\nu K_{\nu,j} \quad (15)$$

has a unique solution in  $K_{\nu, m_\nu}$ . As  $K_{\nu, m_\nu}$  is an  $s_\nu$ -dimensional parameter vector, (15) constrains  $K_{\nu, m_\nu}$  to lie upon an  $(s_\nu - 1)$ -dimensional hyperplane, which has empty interior. Thus the set of parameters  $K$  that lead to a loss of rank in  $X(K)$  is given by the union of at most  $n$  hyperplanes

of empty interior. This set therefore has empty interior, and thus also zero Lebesgue measure. Thus we see that  $X(K)$  and hence  $V(K)$  are non-singular for almost all choices of the parameter matrix  $K$ . ■

The above formulation takes its inspiration from the proof of Proposition 1 in [9], and hence we shall refer to (5)-(8) as the *Moore parametric form* for  $X$  and  $F$ . We note however that [9] only considered the case of distinct eigenvalues, and did not offer any explicit parametric formula for the pole-placing gain matrix. Moreover, it did not show that all matrices  $X$  and  $F$  solving (2) could be parameterized in the above manner.

It is interesting to compare this parametric form with that of [7], in which the eigenvectors comprising  $X$  were obtained from the nullspaces of the matrices  $U_1(A - \lambda_i I)$ , where the parameter  $U_1$  was obtained from the QR-factorization for  $B = [U_0 \ U_1][Z \ 0]^T$ , and was also required to satisfy  $U_1(AX - X\Lambda) = 0$ . By contrast, the Moore parametric form obtains the eigenvectors directly from the nullspaces of the system matrices  $[A - \lambda_i I_n \ B]$ .

### III. ROBUST AND MINIMUM GAIN POLE PLACEMENT

When  $A + BF$  has  $n$  distinct eigenvalues, the sensitivity of an eigenvalue  $\lambda_i$  of  $A + BF$  to perturbations in  $A$ ,  $B$ , and  $F$  can be represented by the condition number [10]

$$c_i = \frac{\|y_i\|_2 \|x_i\|_2}{|y_i^T x_i|} \quad (16)$$

where  $y_i$  and  $x_i$  are the left and right eigenvectors associated with  $\lambda_i$ . For the case where  $A + BF$  is non-defective but has repeated eigenvalues, see [20] for a definition of the corresponding condition numbers. Furthermore, we have [7]

$$c_\infty := \max_i c_i \leq \kappa_2(X) \leq \kappa_{fro}(X) \quad (17)$$

where  $\kappa_2(X) = \|X\|_2 \|X^{-1}\|_2$  and  $\kappa_{fro}(X) = \|X\|_{fro} \|X^{-1}\|_{fro}$  are the condition numbers of the matrix of eigenvectors  $X$  with respect to the Euclidean and Frobenius norms. Following [18], [14], [15], we propose to address the REPP problem by minimizing the condition number of  $X$  with respect to Frobenius norm. The objective function to be minimized is

$$f_1(K) = \kappa_{fro}(X(K)) = \|X(K)\|_{fro} \|X^{-1}(K)\|_{fro} \quad (18)$$

where the input parameter matrix  $K$  is defined as in Proposition 2.1. Note it is possible to reduce the Frobenius norm of a matrix  $X$  by suitably scaling the lengths of its column vectors. When  $X$

is the solution to (2), such scaling does not improve the eigenvalue conditioning in (16). Hence we assume that the column vectors of  $X$  have been normalised.

As pointed out in [14], for efficient computation we can study an alternative objective function

$$f_2(K) = \|X(K)\|_{fro}^2 + \|X^{-1}(K)\|_{fro}^2 \quad (19)$$

because the two objective functions are equivalent. An imported related problem is that of minimizing the norm of the gain matrix  $F$ . The *minimum gain robust exact pole placement problem* (MGREPP) involves simultaneously minimizing both the conditioning and the matrix gain via the weighted objective function

$$f_3(K) = \alpha \kappa_{fro}(X(K)) + (1 - \alpha) \|F(K)\|_{fro} \quad (20)$$

where  $\alpha$  is a weighting factor, with  $0 \leq \alpha \leq 1$ . Minimizing  $f_3$  involves a gradient search employing the first and second order derivatives of  $\kappa_{fro}(X(K))$  and  $\|F(K)\|_{fro}$ ; expressions for these were given in [1].

#### IV. PERFORMANCE COMPARISON OF ROBUST POLE PLACEMENT METHODS

In this section we conduct extensive numerical experiments to compare the performance of our method against those of [7], [18], [13], [14] and [16]. To provide a comprehensive contemporary survey, we implemented these algorithms on the same modern computer, an Intel® Core™ Quad CPU, Model Q9400 at 2.66 GHz with 3326 MB of RAM running Windows™ XP and MATLAB® 2012a. Implementation of [7] was done with MATLAB®'s *place* command. For [13] and [16], we used the *robpo* and *sylvplace* MATLAB® toolboxes, kindly provided to us by the authors. For [14], [18] and our own method, we wrote MATLAB® toolbox implementations for each. The [18] algorithm requires an LMI solver; we chose the public-domain *cvx* toolbox [21]. We shall refer to these as *byersnash*, *rfbt* and *span* (our own method). The names are derived from the names of the respective authors.

To obtain a fair comparison between these methods, we need to consider the runtime allocated to them. The methods of [14], [16] and our proposed method all employ gradient iterative searches, so the values they deliver are contingent upon the initial condition (input parameter matrix  $K$ ) used. The *sylvplace* toolbox randomly generates an initial condition, and thus offers different outputs (different  $F$ ) each time it is run. To obtain repeatable results, we provided



the *byersnash* and *span* toolboxes with a pre-specified collection of input parameter matrices  $K$  composed of canonical vectors. The output shown from each of *byersnash*, *sylvplace* and *span* is the best result from all the initial conditions searched within the allocated runtime. By contrast *place*, *robpole* and *rfbt* all employ a designated starting point, and hence their runtime is simply the time taken to execute their method.

#### A. Robust conditioning comparison using the Byers and Nash benchmark examples

Byers and Nash [14] gave a collection of eleven benchmark example systems, and many authors, including [13], [16] and [18] used these examples to compare the performance of their pole placement methods. Following this well-established tradition, our first set of comparisons employs these well-known examples. The results are given in Table I. We have used  $\kappa_{fro}(X)$  as the performance measure, and we also show the matrix gain used.

The average runtimes for *place*, *robpole* and *rfbt* for the 11 sample systems were 0.05, 0.095 and 14.1 seconds, respectively. For *byersnash*, *sylvplace* and *span* we arbitrarily set the runtime to be  $n$  seconds, where  $n$  is the system dimension, leading to average runtimes of 4.5 seconds, this being the average of the system dimensions in the collection.

Ignoring differences in the conditioning of smaller than 1%, we conclude that *byersnash* and *span* had the best or equal best conditioning in all 11 examples. *sylvplace* and *rfbt* had the best or equal best in 7 cases, while *robpole* had best or equal best in 5 cases. Finally *place* gave the best or equal best in 4 cases. *place* and *robpole* had the shortest runtimes, while *rfbt* had noticeably the longest. We note that the conditioning numbers given here differ significantly from those that were published in [14] and [18]. This may be explained by the fact that these authors did not require the columns of  $X$  to be of unit length. Since methods [7] and [13] normalise the columns of  $X$ , this is essential for a fair comparison of all six methods.

#### B. Robust conditioning comparison with sets of higher-dimensional systems

To probe more deeply into the performance delivered by these six methods, we need to move beyond the low-dimensional examples in the Byers and Nash collection. In Survey 2 we generated three sets of 500 sample systems with  $(A, B)$ , all of state dimension  $n = 20$ , and with control input dimensions of  $m = 2$ ,  $m = 4$  and  $m = 8$ . The pole positions  $\mathcal{L}$  were chosen to be all distinct, with a mixture of real and complex values. The entries of  $A$ ,  $B$  and  $\mathcal{L}$  took

uniformly distributed values within the interval  $[-2, 2]$ . To compare the conditioning, accuracy, and matrix gain of each method, we computed, for each system  $j \in \{1, \dots, 500\}$  and each method  $\star \in \{place, robpole, byersnash, sylvplace, rfbt, span\}$ ,

- $\kappa_{fro}(\star, j)$ : the Frobenius conditioning of method  $\star$  for the  $j$ -th system;
- $c_\infty(\star, j)$ : the  $c_\infty$  conditioning of method  $\star$  for the  $j$ -th system;
- $\Delta(\star, j)$ : the accuracy of method  $\star$  on the  $j$ -th system, equal to the largest absolute value difference between each eigenvalue of  $A + BF$  and the corresponding  $\lambda_i$  in  $\mathcal{L}$ .
- $\|F\|_{fro}(\star, j)$ : the Frobenius norm of  $F$  from Method  $\star$  on system  $j$ .

Noting that *place* is the industry standard for the REPP, we chose to compare all the other methods according to their ability to improve upon *place*, and computed comparative performance indices relative to *place* for each method, and for each performance criterion, as follows:

$$(1 - \text{index}(\star, \kappa_{fro}))^{500} = \prod_{j=1}^{500} \frac{\kappa_{fro}(\star, j)}{\kappa_{fro}(place, j)} \quad (21)$$

$$(1 - \text{index}(\star, c_\infty))^{500} = \prod_{j=1}^{500} \frac{c_\infty(\star, j)}{c_\infty(place, j)} \quad (22)$$

$$(1 - \text{index}(\star, \Delta))^{500} = \prod_{j=1}^{500} \frac{\Delta(\star, j)}{\Delta(place, j)} \quad (23)$$

$$(1 - \text{index}(\star, \|F\|_{fro}))^{500} = \prod_{j=1}^{500} \frac{\|F\|_{fro}(\star, j)}{\|F\|_{fro}(place, j)} \quad (24)$$

For example, in (24), if  $\text{index}(robpole, \|F\|_{fro}) = 0.1$ , then Method *robpole* gives values of  $\|F\|_{fro}$  that are on average 10% smaller than *place*. Larger indices imply greater improvement on *place*, and negative indices indicate performance inferior to *place*. The local gradient search methods *span*, *byersnash* and *sylvplace* were each given 20 seconds of runtime per sample system; the results shown in Table II represent the best conditioning performance achieved from all the initial conditions searched within that time period. For *robpole* and *rfbt*, the average runtime per sample system were 0.552 and 125 seconds ( $m = 2$ ), 0.552 and 82.9 seconds ( $m = 4$ ), and 0.552 and 55.2 seconds ( $m = 8$ ).

The results show that the best performance for robustness and gain minimisation were given by *span*, *byersnash* and *sylvplace*. Both *sylvplace* and *rfbt* were less accurate than *place*, by several orders of magnitude in the case of *rfbt*, which also required substantially longer runtime.

While all methods offered improved conditioning with reduced gain over *place*, this was reduced for the larger values of  $m$ , which may be attributed to the improved performance of *place* when it has more control inputs to work with.

### C. Weighted gain minimisation and conditioning problem

Among the methods in our survey, only [16] (*sylvplace*) considered the MGREPP problem (20). Our Survey 3 compares the performance of *sylvplace* and *span* for the same 500 sample systems used in Survey 2, with  $m = 2$ , for several different values of the weighting factor  $\alpha$ . We again gave *span* and *sylvplace* 20 seconds of runtime per sample system, and computed the performance improvement indices (21)-(24) relative to the gain matrix delivered by *place*; again larger figures indicate greater improvement. The results are shown in Table III. Both methods were able to offer significant reductions in gain, at the price of some reduction in the robustness measures, relative to the pure robustness problem ( $\alpha = 1$ ). However *span* did so with far superior accuracy. Considering the impact of different values of the weighting factor, we see that for  $\alpha = 0.1$ , there was little difference in the conditioning, and only slight improvement in the matrix gain. For  $\alpha \rightarrow 0$  we observed up considerable reduction in the matrix gain, but this eventually comes at the cost of significantly inferior conditioning. These results suggest values around  $\alpha = 0.001$  can give a good balance between these two criteria.

### D. Systems with uncontrollable modes

The EPP problem remains well-posed for systems with uncontrollable modes, provided these are included within the set  $\mathcal{L}$ . The methods *place*, *sylvplace*, *robpole*, *rftb* all assumed controllability of the system, as part of their problem formulation. In principle this involves no loss of generality, since the application of a Householder staircase transformation can decompose any system into its controllable and uncontrollable parts. Nonetheless it is interesting to consider the ability of these toolboxes to accommodate uncontrollable modes. In our final survey, we obtained 100 systems  $(A, B)$ , with  $n = 3$  and  $m = 2$ , that contained one uncontrollable mode. We then chose  $\mathcal{L}$  to include this mode, plus one pair of complex conjugate modes. We defined failure to solve the EPP as being any one of (i) an error was returned upon execution of the algorithm, (ii) any of the closed-loop poles differed by more than 5% from their desired location, and (iii) the gain of  $F$  was undefined or greater than  $10^{10}$ . We observed failures as follows: *place*,

*sylvplace*, *robpole* and *rfbt* had 100, 98, 30 and 12 failures, respectively; we concluded these toolboxes in their present form cannot reliably solve the EPP in these conditions. *byersnash* and our method *span* had no failures; we attribute their superior reliability to their usage of nullspace methods. Uncontrollable modes increase the column dimension of the corresponding nullspace basis matrix; for *byersnash* and *span* this is readily accommodated by adjusting the row dimension of the parameter matrix.

## V. CONCLUSION

We have introduced a parametric formula for the exact pole placement of linear systems via state feedback, derived from Moore's classic eigenstructure method. This parametric form was used to formulate the robust and minimum gain exact pole placement problem as an unconstrained optimization problem, to be solved by gradient iterative methods.

The method was implemented as a MATLAB<sup>®</sup> toolbox called *span*, and its performance was compared against several other methods from the classic and recent literature. All methods considered gave superior performance to the widely used MATLAB<sup>®</sup> `place` command, albeit with somewhat longer runtime. When the Frobenius conditioning of the eigenvector matrix is used as the robustness measure, the best performance was provided by the our proposed method, and also the Byers-Nash method. The results suggest that, in comparison with heuristic methods, gradient iterative methods are best able to take advantage of the high levels of computational power that are now widely available. They also suggest that methods based on nullspaces of appropriate system matrices may offer superior accuracy of pole placement to those adopting Sylvester matrix transformations.

For a given system  $(A, B, \mathcal{L}, \mathcal{M})$ , *byersnash* and *span* will in general yield quite different gain matrices, offering different performance values, so both methods should be considered for optimal performance. While Byers and Nash considered only the robustness, our method is able to accommodate a combined robustness and gain minimization approach, enabling the designer to obtain significantly reduced gain in exchange for somewhat inferior conditioning.

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TABLE I  
SURVEY 1: REPP WITH THE BYERS NASH EXAMPLES

Example	<i>place</i> [7]		<i>byersnash</i> [14]		<i>robpole</i> [13]	
	$\kappa_{fro}(X)$	$\ F\ _{fro}$	$\kappa_{fro}(X)$	$\ F\ _{fro}$	$\kappa_{fro}(X)$	$\ F\ _{fro}$
1	6.5641	1.364	6.4451	1.4582	7.3214	1.3338
2	57.491	301.37	50.224	355.19	52.972	224.95
3	103.18	105.06	46.238	77.215	55.987	49.104
4	13.431	9.899	13.421	9.4485	13.421	9.4462
5	146.18	4.8496	142.39	4.5561	144.78	5.4168
6	6.0018	21.5	5.9633	23.25	6.0262	20.197
7	12.375	233.64	11.302	326.35	12.017	235.08
8	36.986	15.7600	6.1824	28.033	6.1824	28.599
9	28.682	2356.5	23.915	832.22	23.937	823.70
10	4.0029	1.4897	4.113	5.2687	4	1.5174
11	14618	6692.1	14510	6580.8	14510	6580.7

  

Example	<i>sylvplace</i> [4]		<i>rft</i> [18]		<i>span</i>	
	$\kappa_{fro}(X)$	$\ F\ _{fro}$	$\kappa_{fro}(X)$	$\ F\ _{fro}$	$\kappa_{fro}(X)$	$\ F\ _{fro}$
1	6.5997	1.4662	6.5595	1.5253	6.4451	1.4582
2	50.042	327.75	50.185	361.01	50.224	355.17
3	45.741	72.285	45.772	73.582	46.223	77.146
4	13.421	9.4465	13.421	9.366	13.421	9.4432
5	141.99	4.8472	142.82	4.3963	142.39	4.556
6	5.9361	22.474	6.4086	14.771	5.9622	23.318
7	11.353	271.17	12.280	297.85	11.301	271.06
8	6.1824	21.827	9.381	39.300	6.1824	21.102
9	24.23	903.11	23.925	884.84	23.916	831.23
10	4.113	1.513	4	1.5185	4	1.517
11	16571	10716	14475	6642	14510	6581.3

TABLE II  
SURVEY 2: REPP WITH HIGHER-DIMENSIONAL SYSTEMS

System Dimension	Metric	<i>byersnash</i> [14]	<i>robpole</i> [13]	<i>sylvplace</i> [4]	<i>rfbt</i> [18]	<i>span</i>
$n = 20,$ $m = 2,$ $\text{sys} = 500$	$\kappa_{fro}(X)(\%)$	54.670	9.8815	51.938	41.332	54.603
	$c_{\infty}(\%)$	62.047	10.620	59.759	49.447	61.983
	$\ F\ _{fro}(\%)$	23.555	1.9292	22.310	14.337	23.276
	Accuracy (%)	67.356	26.998	-1.0082	-46237	64.344
$n = 20,$ $m = 4,$ $\text{sys} = 500$	$\kappa_{fro}(X)(\%)$	37.268	9.150	36.725	31.048	37.264
	$c_{\infty}(\%)$	49.418	9.8601	50.226	43.374	49.400
	$\ F\ _{fro}(\%)$	15.677	4.3745	15.524	11.163	15.698
	Accuracy (%)	45.057	23.760	-65.586	-169100	43.034
$n = 20,$ $m = 8,$ $\text{sys} = 500$	$\kappa_{fro}(X)(\%)$	15.198	7.7702	11.745	12.849	15.197
	$c_{\infty}(\%)$	23.271	10.067	20.848	20.840	23.236
	$\ F\ _{fro}(\%)$	3.7940	4.7471	3.3979	1.7034	3.7860
	Accuracy (%)	18.525	17.8859	-44.635	-338240	16.225

TABLE III  
SURVEY 3: MGREPP WITH HIGHER DIMENSIONAL SYSTEMS ( $n = 20, m = 2, \text{sys} = 500$ )

Metric	$\alpha = 0.0001$		$\alpha = 0.001$		$\alpha = 0.1$	
	<i>span</i>	<i>sylvplace</i> [4]	<i>span</i>	<i>sylvplace</i> [4]	<i>span</i>	<i>sylvplace</i> [4]
$\kappa_{fro}(X)(\%)$	-25.578	23.980	37.641	41.906	53.936	51.699
$c_{\infty}(\%)$	-13.540	33.929	45.966	51.379	61.213	59.465
$\ F\ _{fro}(\%)$	50.319	38.046	43.577	37.740	27.509	26.404
Accuracy (%)	16.992	-46.326	57.833	-16.025	65.643	-1.0463